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DEPARTMENT OF CIVIL ENGINEERING
AND ENGINEERING MECHANICS

AN EXTENSION OF THE THEORY OF WATER HAMMER

by

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Richard S

AFSWP No. 922
Office of Naval Research
Contract No. Nonr-266(08)
Technical Report No. 15
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AN EXTENSION OF THE THEORY OF WATER HAMMER

By R. Skalak

NOMENCLATURE

The following nomenclature is used in the paper:

c	Velocity of sound in fluid
c_e	Joukowsky water hammer velocity (See eq. 1)
K	Bulk modulus of fluid
a	Radius of tube
h	Thickness of tube wall
E	Young's modulus of tube wall material
p	Pressure in fluid
v	Axial velocity of fluid
z	Distance axially along the tube
t	Time
u	Axial displacement of tube
w	Radial deflection of tube wall (+ outward)
ρ_o	Initial density of fluid
r	Radial distance
ϕ	Velocity potential
m	Mass of tube per unit surface area
ν	Poisson's ratio for tube wall
c_o	$\sqrt{\frac{Eh}{m(1-\nu^2)}}$ Velocity of sound in tube wall
q	Force per unit area on tube wall
ξ, Ω	Transform variables
n, s	Integers
N	See eq. (23)
Δ	See eq. (24)
J_o, J_1	Bessel functions
i =	$\sqrt{-1}$
D	See eq. (30)
F	See eq. (36)
G	See eq. (37)
B	See eq. (38)

$c', \bar{c}$	Arbitrary constant velocities	I, H	Integrals. See eq. (53)
c_1, c_2	Phase velocities	L	Length of wave front
z'	See eq. (48)	R	$= \frac{c_o^2}{c^2}$
η	Variable of integration		
β	See eq. (50	A	$= \frac{\rho_o a}{m}$

1. Introduction

The usual theory of water hammer which is used to compute the magnitude and velocity of pressure waves in a pipe filled with an elastic fluid was given by Joukowsky [8][1] in 1898. This theory predicts that pressure waves travel without change of shape at a velocity c_e:

$$C_e = \frac{C}{\sqrt{1 + \frac{2Ka}{Eh}}} \tag{1}$$

In the derivation of this formula it is assumed that the pressure is uniform across any section of the pipe and that the deflection of the pipe is equal to the static deflection due to the instantaneous pressure in the fluid. The first assumption neglects, in effect, the inertial forces associated with radial motion of the fluid. The second neglects the mass of the pipe wall and the longitudinal and bending stresses in the pipe wall.

For the case of an infinite train of sinusoidal pressure waves in a tube filled with fluid, much work has been done without resorting to the above assumptions. In 1878 Korteweg [9] showed that the phase velocity of sinusoidal waves varies with the wave-length when the radial inertia of the fluid and the mass of the pipe wall are taken into account. Korteweg also gave the expression for c_e, eq. (1) as an approximation of the phase velocity for long wave-lengths. The work of Korteweg was extended in 1898 by Lamb [10] who considered the pipe as an elastic membrane thus including the effect of longitudinal stresses in the pipe wall. Lamb showed there are two finite phase velocities as the wave length approaches infinity. One of these is very close to Korteweg's result c_e and the other is very nearly the velocity of sound in the material of the tube wall. Recent papers by Jacobi [6] and Thomson [14] include the flexural stiffness of the pipe wall. The inclusion of the bending stresses affects the phase velocities appreciably only for very short wave-lengths.

[1] Numbers in brackets refer to the Bibliography at the end of the paper.

In the present paper water hammer waves are considered without some of the simplifications made in the Joukowsky theory. The equations of motion for a thin cylindrical tube are used, cf. Flügge [11], with one additional term for the effect of rotatory inertia. The fluid is assumed to be elastic and inviscid but no assumptions are made regarding its motion except that it is small.

Only the case where the velocity of sound in the material of the pipe wall is greater than in the fluid will be considered. Much of the analysis would be valid in the opposite case, but the relative speeds of the various waves would differ.

2. A Formal Solution by Integral Transforms

In this section a formal solution of a typical water hammer problem is obtained in the form of a double integral. The solution is reduced to terms of single real integrals whose evaluation is discussed in the next section.

The problem considered is to find the motion of a thin cylindrical tube filled with fluid starting from given initial conditions. The conditions, shown in Fig. 1, are that for $t = 0$:

$$\left.\begin{array}{l} p = 0,\ v = 0 \text{ for } z > 0 \\ p = p_o,\ v = v_o \text{ for } z < 0 \\ u = 0,\ w = 0, \text{ for all } z \end{array}\right\} \tag{2}$$

The velocity v_o is assumed to be related to the pressure p_o by:

$$v_o = \frac{p_o}{\rho_o c} \tag{3}$$

This is the velocity that would exist behind a plane shock wave of pressure p_o in an infinite body of fluid. Thus the initial conditions for the fluid correspond to a step shock wave moving in the positive direction. An external pressure equal to p_o is assumed acting on the outside of the tube for $z < 0$ to prevent any motion of the tube at $z = -\infty$. This external pressure is assumed to remain on the tube for $z < 0$ for all t.

These initial conditions are intended to simulate the situation arising when a pipe leading from a reservoir is filled with fluid at rest below reservoir pressure and a valve at the reservoir end is suddenly opened. The part of the tube

left of the origin in Fig. 1 is a convenient device for supplying fluid and energy in place of a reservoir. It is expected that the particular means of generating a water hammer wave will not essentially affect its characteristics when the wave has progressed far enough down the pipe.

The motion of the fluid will be described by a velocity potential which must satisfy the equation:

$$\nabla^2 \phi = \phi_{rr} + \frac{1}{r}\phi_r + \phi_{zz} = \frac{1}{c^2}\ddot{\phi} \tag{4}$$

where the subscripts r and z denote derivatives.

The motion of the tube due to a radial force per unit area, q (z, t), is governed by two equations:

$$\frac{\ddot{w}}{c_o^2} - \frac{h^2}{12c_o^2}\ddot{w}_{zz} + \left(1 + \frac{h^2}{12a^2}\right)\frac{w}{a^2} + \frac{\nu}{a}u_z + \frac{h^2}{12}w_{zzzz} - \frac{h^2}{12a}u_{zzz} = \frac{q}{mc_o^2} \tag{5}$$

$$\frac{\ddot{u}}{c_o^2} - u_{zz} - \frac{\nu}{a}w_z + \frac{h^2}{12a}w_{zzz} = 0 \tag{6}$$

A detailed derivation of these equations is given in [12]. They agree with the equations given by Flügge [11] except that the rotatory inertia term $\ddot{w}_{zz}$ has been added above.

The fact that the fluid and tube are expected to remain in contact is expressed by:

$$\dot{w} = -\left[\frac{\partial\phi}{\partial r}\right]_{r=a} \tag{7}$$

The pressure exerted on the tube wall by the fluid is:

$$\left[p\right]_{r=a} = \rho_o\left[\frac{\partial\phi}{\partial t}\right]_{r=a} \tag{8}$$

This is the value of q to be used for $z > 0$ in eq. (5). For $z < 0$, the external pressure $- p_o$ must be added.

It is convenient to consider the solution of the problem as the sum of two parts:

The first part is the shock wave that would be produced by the initial conditions if the tube were rigid. The tube displacements w_1, u_1, for this part are zero. The velocity and pressure v_1, p_1, for the fluid are:

$$\left.\begin{array}{l} v_1 = v_o \\ p_1 = p_o \end{array}\right\} \text{ for } z < ct \qquad\qquad \left.\begin{array}{l} v_1 = 0 \\ p_1 = 0 \end{array}\right\} \text{ for } z > ct \tag{9}$$

The second part is the response of the tube and fluid to a progressive load pressure p_s equal to the pressure produced by the shock wave of the first part. This load pressure acts on the tube wall radially outward. Its magnitude is p_o for $0 < z < ct$ and zero elsewhere.

Inasmuch as w_1, u_1, v_1, etc. are known, the problem reduces to finding the the response w_2, u_2, v_2, etc. of the tube and fluid to the load pressure p_s. For this second part of the solution the system is initially at rest and the governing equations are (4), (5), (6), (7) with:

$$q_2 = \rho_o \left[\frac{\partial \phi}{\partial t}\right]_{r=a} + p_s \tag{10}$$

The solution is begun by transforming the governing equations and eliminating the velocity potential ϕ_2. The remaining transformed equations are solved simultaneously and the inversions of the transformations then yield the solution.

With respect to z a Fourier transform, cf. [15] p. 42, is used. Defining the transform of a function g (z, t) with respect to z by:

$$\bar{g}(\xi,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(z,t)\, e^{-i\xi z}\, dz \tag{11}$$

then the inversion is:

$$g(z,t) = \int_{-\infty}^{\infty} \bar{g}(\xi,t)\, e^{i\xi z}\, d\xi \tag{12}$$

With respect to time a Laplace transform is used, but the transform variable adopted is $-i = -\sqrt{-1}$ times the usual transform variable. This results in a certain symmetry with the first transform and convenience in later analysis. Defining the

transform of a function $\bar{g}$ (ξ , t) with respect to t by:

$$\bar{\bar{g}}(\xi,\Omega)=\frac{1}{2\pi}\int_0^\infty \bar{g}(\xi,t)\,e^{-i\Omega t}\,dt \tag{13}$$

then the inversion is:

$$\bar{g}(\xi,t)=\int_{-\infty-i\alpha}^{\infty-i\alpha}\bar{\bar{g}}(\xi,\Omega)\,e^{i\Omega t}\,d\Omega \tag{14}$$

The double transform of derivatives of g (z, t) may be expressed in terms of the double transform of g (z, t) by the usual partial integration procedure, [13] p. 27. Using the fact that initial velocities and displacements are zero, the transforms of the derivatives involved are:

$$\overline{\overline{\frac{\partial^n}{\partial z^n}\frac{\partial^s}{\partial t^s}\left[g(z,t)\right]}}=(i\xi)^n(i\Omega)^s\,\bar{\bar{g}}(\xi,\Omega) \tag{15}$$

Using q_2 as given by eq. (10), the double transformation of eqs. (4), (5), (6) and (7) yields:

$$\bar{\bar{\phi}}_{2rr}+\frac{1}{r}\bar{\bar{\phi}}_{2r}-\xi^2\bar{\bar{\phi}}_2=-\frac{\Omega^2}{c^2}\bar{\bar{\phi}}_2 \tag{16}$$

$$-\frac{\Omega^2}{c_o^2}\bar{\bar{w}}_2-\xi^2\Omega^2\frac{h^2}{12c_o^2}\bar{\bar{w}}_2+\left(1+\frac{h^2}{12a^2}\right)\frac{\bar{\bar{w}}_2}{a^2}+i\xi\frac{\nu}{a}\bar{\bar{u}}_2+\frac{\xi^2h^4}{12}\bar{\bar{w}}_2+i\xi^3\frac{h^2}{12a}\bar{\bar{u}}_2=$$

$$=i\Omega\frac{\rho_o}{mc_o^2}\left[\bar{\bar{\phi}}_2\right]_{r=a}+\frac{\bar{\bar{p}}_s}{mc_o^2} \tag{17}$$

$$-\frac{\Omega^2}{c_o^2}\bar{\bar{u}}_2+\xi^2\bar{\bar{u}}_2-i\xi\frac{\nu}{a}\bar{\bar{w}}_2-i\xi^3\frac{h^2}{12a}\bar{\bar{w}}_2=0 \tag{18}$$

$$i\Omega\bar{\bar{w}}_2=-\left[\bar{\bar{\phi}}_{2r}\right]_{r=a} \tag{19}$$

The double transforms are functions of ξ and Ω only except for $\bar{\bar{\phi}}_2$ which is $\bar{\bar{\phi}}_2$ (ξ, Ω, r). From eq. (16) it is found that the r dependence of $\bar{\bar{\phi}}_2$ involves a Bessel function of the first kind. The function p_s is known. Solving eqs. (16), (17), (18), (19 for $\bar{\bar{w}}_2$, $\bar{\bar{u}}_2$, $\bar{\bar{\phi}}_2$ and inverting yields:

$$w_2 = -\frac{p_o}{4\pi^2 m}\int_{-\infty}^{\infty} e^{i\xi z}\,d\xi \int_{-\infty-i\alpha}^{\infty-i\alpha} \frac{e^{i\Omega t}}{\Omega^3\left(\frac{\Omega}{c}+\xi\right)\left[\frac{\rho_o a}{2m}\Delta+\frac{1}{N}\right]}\,d\Omega \tag{20}$$

$$u_2 = \frac{p_o}{4\pi^2 m}\int_{-\infty}^{\infty} e^{i\xi z}\,d\xi \int_{-\infty-i\alpha}^{\infty-i\alpha} \frac{\nu i \xi\left(1+\frac{h^2}{12\nu}\xi^2\right)e^{i\Omega t}}{a\left(\frac{\Omega^2}{c_o^2}-\xi^2\right)\Omega^3\left(\frac{\Omega}{c}+\xi\right)\left[\frac{\rho_o a}{2m}\Delta+\frac{1}{N}\right]}\,d\Omega \tag{21}$$

$$\phi_2 = -\frac{p_o}{4\pi^2 m}\int_{-\infty}^{\infty} e^{i\xi z}\,d\xi \int_{-\infty-i\alpha}^{\infty-i\alpha} \frac{i\,J_o\left(r\sqrt{\frac{\Omega^2}{c^2}-\xi^2}\right)e^{i\Omega t}\,d\Omega}{\Omega^2\sqrt{\frac{\Omega^2}{c^2}-\xi^2}\;J_1\left(a\sqrt{\frac{\Omega^2}{c^2}-\xi^2}\right)\left(\frac{\Omega}{c}+\xi\right)\left[\frac{\rho_o a}{2m}\Delta+\frac{1}{N}\right]} \tag{22}$$

where:

$$\frac{1}{N} = -1+\frac{\Omega^2-c_o^2(1-\nu^2)\xi^2}{\Omega^2 a^2\left(\frac{\Omega^2}{c_o^2}-\xi^2\right)}+\frac{h^2}{12}\left[-\xi^2+\frac{c_o^2}{a^4\Omega^2}+\frac{2c_o^2\nu\xi^4}{\Omega^2 a^2\left(\frac{\Omega^2}{c_o^2}-\xi^2\right)}+\frac{c_o^2\xi^4}{\Omega^2}+\frac{c_o^2 h^2\xi^6}{12\Omega^2 a^2\left(\frac{\Omega^2}{c_o^2}-\xi^2\right)}\right] \tag{23}$$

$$\Delta = \frac{2\,J_o\left(a\sqrt{\frac{\Omega^2}{c^2}-\xi^2}\right)}{a\sqrt{\frac{\Omega^2}{c^2}-\xi^2}\;J_1\left(a\sqrt{\frac{\Omega^2}{c^2}-\xi^2}\right)} \tag{24}$$

The evaluation of the radial deflection w_2 will be considered first. It is typical of the quantities of interest. The inner or Ω integral of eq. (20) may be evaluated by Cauchy's residue theorem applied in the complex Ω plane. During this process ξ is regarded as a constant real parameter since the outer integral involves only real values of ξ.

The factor Ω^3 in the denominator of the integrand in eq. (20) produces a simple pole at $\Omega = 0$ because $(\Omega^2 \frac{1}{N})$ is finite at $\Omega = 0$. There is no pole at $(\frac{\Omega}{c}+\xi) = 0$ because $(\frac{\Omega}{c}+\xi)\Delta$ is finite at this point. There are poles at the points for which the remaining bracket is zero, i.e., where:

$$\left[\frac{\rho_o a}{2m}\Delta+\frac{1}{N}\right] = 0 \tag{25}$$

For any real ξ this equation has an infinite number of roots Ω_n. These roots have an interesting physical significance. They are the circular frequencies of the modes of free vibration of the tube and fluid system having a wave-length $\lambda = \frac{2\pi}{\xi}$. Alternatively the ratios $\frac{\Omega_n}{\xi}$ are the phase velocities c_n of progressive sinusoidal waves of wave-length $\frac{2\pi}{\xi}$. These facts may be developed by considering the governing equations (4), (5), (6) and (7) with q in eq. (5) equal to the fluid pressure only. A solution is assumed of the form:

$$\left.\begin{aligned} w &= w' e^{i(\xi z + \Omega t)} \\ u &= u' e^{i(\xi z + \Omega t)} \\ \phi &= \phi' f(r) e^{i(\xi z + \Omega t)} \end{aligned}\right\} \qquad (26)$$

where the prime quantities are constants and f(r) is an unknown function of r. Substituting eqs. (26) in eqs. (4), (5), (6) and (7) the constants and f(r) may be eliminated, leading to exactly the transcendental eq. (25) relating ξ and Ω.

For real values of ξ all the roots Ω_n of eq. (25) are also real. If any of the Ω_n were complex or imaginary the corresponding free vibration would grow or decay indefinitely in time. Since the system has no means of dissipating or radiating energy, this is not possible physically.

Fig. 2 shows the Ω_n vs ξ values for a typical case in which the velocity of sound in the tube wall is greater than in the fluid, i.e., $c_o > c$. It is found that for any value of ξ there is one root Ω_1, for which $(\frac{\Omega^2}{c^2} - \xi^2)$ is negative. This root corresponds to a phase velocity less than c, the velocity of sound in the fluid. There are also an infinite number of roots for which $(\frac{\Omega^2}{c^2} - \xi^2)$ is positive. These correspond to phase velocities greater than c. Fig. 3 shows the same data as in Fig. 2 plotted as phase velocity vs wave-length.

The fact that two sets of Ω_n curves pass through the origin in Fig. 2 shows that there are two modes which have finite phase velocities as the wave length approaches infinity and the frequency approaches zero. This was also shown by Lamb [10], p. 7. It is mentioned here because in a recent paper [14], p. 933, it is erroneously stated that there is a cutoff frequency below which only one mode may be propagated.

The above discussion shows that the poles of the integrand of eq. (20) are at $\Omega = 0$ and $\Omega = \Omega_n$, all on the real Ω axis. The path of integration is below this axis and it will be closed in the usual manner by adding a semicircle of infinite radius in the upper half of the Ω plane. The integral over the semicircle approaches zero as the radius approaches infinity. The contribution of the pole at $\Omega = 0$ to w_2 is:

$$w_{2a} = \frac{p_o}{2\pi i m}\int_{-\infty}^{\infty} \frac{e^{i\xi z}}{\xi\left[\frac{c_o^2}{a^2}\left(1-\nu^2+\frac{h^2}{12a^2}\right)-\frac{c_o^2 \nu h^2}{6a^2}\xi^2+\frac{h^2c_o^2}{12}\left(1-\frac{h^2}{12a^2}\right)\xi^4\right]}\,d\xi \qquad (27)$$

The contributions of the Ω_n poles to w_2 may be expressed as:

$$w_{2b} = -\frac{p_o}{4\pi^2 m}\int_{-\infty}^{\infty} e^{i\xi z}\, 2\pi i \sum_{\Omega_n}\left[\frac{e^{i\Omega t}}{\Omega^3\left(\frac{\Omega}{c}+\xi\right)\frac{d}{d\Omega}\left[\frac{\rho_o a}{2m}\Delta+\frac{1}{N}\right]}\right]_{\Omega=\Omega_n} d\xi \qquad (28)$$

The sum indicated is over the Ω_n roots for each value of ξ. The values of the Ω_n as functions of ξ are given by Fig. 2. Eq. (28) may be written in the form

$$w_{2b} = \frac{-p_o}{2\pi i m}\int_{-\infty}^{\infty}\sum_{\Omega_n}\frac{e^{i(\xi z+\Omega t)}}{\left(\frac{\Omega}{c}+\xi\right)D}\,d\xi \qquad (29)$$

where
$$D = -\Omega^3\frac{d}{d\Omega}\left[\frac{\rho_o a}{2m}\Delta+\frac{1}{N}\right] = \Omega^3\left\{\frac{\rho_o\Omega a}{mc^2\left(\frac{\Omega^2}{c^2}-\xi^2\right)}\left[1+\frac{J_o^2}{J_1^2}\right]+\right.$$
$$\left.+\frac{2C_o^2}{\Omega^3a^2}\left[\frac{\nu^2}{\left(1-C_o^2\frac{\xi^2}{\Omega^2}\right)^2}+1-\nu^2+\frac{h^2(1+a^4\xi^4)}{12a^2}+\frac{h^2\xi^4\left(2\frac{\Omega^2}{C_o^2}-\xi^2\right)}{12\left(\frac{\Omega^2}{C_o^2}-\xi^2\right)^2}\left(2\nu+\frac{h^2\xi^2}{12}\right)\right]\right\} \qquad (30)$$

The integral in eq. (29) covers all four quadrants of the Ω_n curves in Fig. 2, but it may be expressed as an integral over the first quadrant values only because D is even in Ω and even in ξ and the Ω_n curves are symmetric about both axes. Thus:

$$w_{2b} = \frac{p_o}{\pi m}\int_0^{\infty}\sum_{\Omega_n}\left[\frac{\sin(\xi z-\Omega t)}{\left(\frac{\Omega}{c}-\xi\right)D}-\frac{\sin(\xi z+\Omega t)}{\left(\frac{\Omega}{c}+\xi\right)D}\right]d\xi \qquad (31)$$

where the summation is over the Ω_n curves in only the first quadrant of Fig. 2. This concludes the formal solution for w. The evaluation of the integrals remaining in eqs. (27) and (31) will be considered in the next section.

Instead of evaluating u_2 and ϕ_2, certain quantities of interest which are

derivatives of u_2 and ϕ_2 will be computed directly. These are:

$$\left.\begin{array}{ll} \text{Pressure in the fluid:} & p_2 = \rho_0 \dfrac{\partial \phi_2}{\partial t} \\ \text{Axial velocity of the fluid:} & v_2 = -\dfrac{\partial \phi_2}{\partial t} \\ \text{Longitudinal strain in the tube:} & u_{z2} = \dfrac{\partial u_2}{\partial z} \end{array}\right\} \tag{32}$$

Expressions for p_2, v_2 and u_{z2} are derived from the double integrals for u_2 and ϕ_2, eqs. (21) and (22), through application of eq. (15). The inner integral in Ω may be evaluated in each case by using Cauchy's residue theorem as in the case of w_2. The results are:

$$p_2 = \frac{p_0}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{i\xi(z-ct)}}{\xi}\, d\xi + \frac{p_0 \rho_0}{\pi m}\int_0^{\infty} \sum_{\Omega_n} \left[\frac{\sin(\xi z - \Omega t)}{\left(\frac{\Omega}{c} - \xi\right) F} - \frac{\sin(\xi z + \Omega t)}{\left(\frac{\Omega}{c} + \xi\right) F} \right] d\xi \tag{33}$$

$$v_2 = \frac{p_0}{2\pi i \rho_0 c}\int_{-\infty}^{\infty} \frac{e^{i\xi(z-ct)}}{\xi}\, d\xi + \frac{p_0}{\pi m}\int_0^{\infty} \sum_{\Omega_n} \left[\frac{\sin(\xi z - \Omega t)}{\left(\frac{\Omega}{c} - \xi\right) G} + \frac{\sin(\xi z + \Omega t)}{\left(\frac{\Omega}{c} + \xi\right) G} \right] d\xi \tag{34}$$

$$u_{z2} = \frac{i p_0 \nu}{2\pi m a}\int_{-\infty}^{\infty} \frac{\left(1 + \frac{h^2 \xi^2}{12\nu}\right) e^{i\xi z}}{f(\xi)}\, d\xi + \frac{p_0 \nu}{\pi m a}\int_0^{\infty} \sum_{\Omega_n} \left[\frac{\sin(\xi z - \Omega t)}{\left(\frac{\Omega}{c} - \xi\right) B} - \frac{\sin(\xi z + \Omega t)}{\left(\frac{\Omega}{c} + \xi\right) B} \right] d\xi \tag{35}$$

where:

$$F = -\frac{D}{\Omega^2}\sqrt{\frac{\Omega^2}{c^2} - \xi^2}\; \frac{J_1\left(a\sqrt{\frac{\Omega^2}{c^2} - \xi^2}\right)}{J_0\left(r\sqrt{\frac{\Omega^2}{c^2} - \xi^2}\right)}. \tag{36}$$

$$G = F\frac{\Omega}{\xi} \tag{37}$$

$$B = +\frac{D\left(\frac{\Omega^2}{C_0^2} - \xi^2\right)}{\xi^2\left(1 + \frac{h^2}{12\nu}\xi^2\right)} \tag{38}$$

and $f(\xi)$ stands for the denominator of the integrand in eq. (27).

The solution of the problem originally posed at the beginning of this section is given formally by:

$$\left.\begin{array}{ll} w = w_{2a} + w_{2b} & \text{Eqs. (27) and (31)} \\ p = p_1 + p_2 & \text{Eqs. (9) and (33)} \\ v = v_1 + v_2 & \text{Eqs. (9) and (34)} \\ u_z = u_{z2} & \text{Eq. (35)} \end{array}\right\} \qquad (39)$$

3. Evaluation of the Solution for Large Values of $|z|$ and t.

In the formal solution indicated by eqs. (39) the integrations with respect to ξ would be difficult to carry out in general because the Ω_n values are known as functions of ξ only through the transcendental equation (25). However, some information may be obtained by considering approximations of the integrals for large values of $|z|$ and t.

Consider the part of w given by eq. (31) and, in particular, values of z and t such that:

$$z = c' t \qquad (40)$$

where c' is an arbitrarily selected velocity. Substituting in eq. (31):

$$W_{2b} = \frac{p_o}{\pi m} \int_0^\infty \sum_{\Omega_n} \left[\frac{\sin(\xi c' - \Omega)t}{\left(\frac{\Omega}{c} - \xi\right) D} - \frac{\sin(\xi c' + \Omega)t}{\left(\frac{\Omega}{c} + \xi\right) D} \right] d\xi \qquad (41)$$

It is found by inspection of D, that the functions $\dfrac{1}{(\frac{\Omega}{c} - \xi)\, D}$ and $\dfrac{1}{(\frac{\Omega}{c} + \xi)\, D}$ are bounded for all of the Ω_n curves for all ξ except for the lowest two curves, Ω_1 and Ω_2, at the point $\xi = 0$. The distinction between the lowest two and the remaining Ω_n curves arises because only the Ω_1 and Ω_2 curves pass through the origin. A physical interpretation of this fact is that only these two lowest modes have finite phase velocities as the wave-length increases indefinitely.

For the higher Ω_n branches, $n > 2$, the integral (41) may be approximated by the method of stationary phase [7] p. 505, for large values of t. This method shows

these integrals to be of the order of $\frac{1}{\sqrt{t}}$.

The integrals over the Ω_1 and Ω_2 curves will be broken into two parts; one part from $\xi = 0$ to $\xi = \epsilon$ and a second part from ϵ to ∞, where ϵ is a small positive number. By the stationary phase theorem the portions from ϵ to ∞ are again $O\left(\frac{1}{\sqrt{t}}\right)$. Collecting these results, eq. (41) becomes:

$$W_{2b} = \frac{P_o}{\pi m} \int_0^{\epsilon} \sum_{\Omega_{1,2}} \left[\frac{\sin(\xi c' - \Omega)t}{\left(\frac{\Omega}{c} - \xi\right) D} - \frac{\sin(\xi c' + \Omega)t}{\left(\frac{\Omega}{c} + \xi\right) D} \right] d\xi + O\left(\frac{1}{\sqrt{t}}\right) \tag{42}$$

Since ϵ may be chosen indefinitely small, the functions in the integrand may be approximated by an appropriate number of terms of their expansions about $\xi = 0$. Consider the first term of the integral with $\Omega = \Omega_1$ which may be written:

$$W_I = \frac{P_o}{\pi m} \int_0^{\epsilon} \frac{\sin(\xi c' - \Omega_1)t}{\xi\left(\frac{\Omega_1}{\xi c} - 1\right) D} d\xi \tag{43}$$

The denominator of the integrand approaches a constant times ξ as ξ approaches zero. The constant is:

$$\lim_{\xi \to 0} \left(\frac{\Omega_1}{\xi c} - 1\right) D = \left(\frac{c_1}{c} - 1\right) D_1 \tag{44}$$

where:

$$\left.\begin{aligned} c_1 &= \lim_{\xi \to 0} \left(\frac{\Omega_1}{\xi}\right) \\ D_1 = \lim_{\xi \to 0} D(\Omega_1, \xi) &= \frac{4 \rho_o c_1^4}{m c^2 a \left(\frac{c_1^2}{c^2} - 1\right)^2} + \frac{2 c_o^2}{a^2} \left[\frac{\nu^2}{\left(1 - \frac{c_o^2}{c_1^2}\right)^2} + 1 - \nu^2 + \frac{h^2}{12 a^2} \right] \end{aligned}\right\} \tag{45}$$

The constant c_1 is the phase velocity of the slowest waves for wave-lengths approaching infinity. It is very close to the Joukowsky velocity c_e. An expression for c_1 is derived in the appendix.

In the numerator of eq. (43) Ω_1 is replaced by the first two terms of its expansion about $\xi = 0$:

$$\Omega_1 = c_1 \xi - d_1 \xi^3 \tag{46}$$

This expansion is developed in the appendix. It is necessary to use two terms of the expansion because the first term alone would result in a vanishing numerator for the case $c' = c_1$.

Using eqs. (44) and (46) in (43) yields:

$$w_I = \frac{p_o}{\pi m\left(\frac{c_1}{c} - 1\right) D_1} \int_0^\infty \frac{\sin(\xi z' + d_1 \xi^3 t)}{\xi} d\xi \tag{47}$$

where z' is defined by:

$$z' = (c' - c_1)\, t \tag{48}$$

The new coordinate z' is the location of any point relative to an origin moving at velocity c_1. In eq. (47) the upper limit has been changed from ϵ to ∞. The integral thus added is $O(\frac{1}{\sqrt{t}})$ by the stationary phase theorem, and is negligible in the present approximation.

For $z' = 0$, the integral in eq. (47) has a constant value of $\frac{\pi}{6}$ provided d_1 is positive which is the case here. For points where $|z'|$ is large, the term containing ξ^3 may be neglected. Then the integral in eq. (47) becomes $\pm \frac{\pi}{2}$ for $\pm z'$. For points near $z' = 0$ it is convenient to change the variable of integration from ξ to $\eta = \xi z'$. Then:

$$w_I = \frac{p_o}{\pi m\left(\frac{c_1}{c} - 1\right) D_1} \int_0^\infty \frac{\sin(\eta + \beta \eta^3)}{\eta} d\eta \tag{49}$$

where:

$$\beta = \frac{d_1 t}{z'^3} \tag{50}$$

From eq. (49) it may be seen that for w_I constant β must be constant, say β_o. Then:

$$z' = \sqrt[3]{d_1 t \beta_o} \tag{51}$$

This shows that points for which w_I is constant continually move away from the point $z' = 0$. To summarize, w_I is a wave having constant deflection at the point $z' = 0$ i.e., moving with velocity c_1. At points far ahead and far behind this wave front point the deflection is also constant, but the wave front continually spreads out

around the point $z' = 0$. Fig. 4 shows the general shape of the wave.

If the same reasoning that has been used to evaluate w_I is applied to the remaining terms of eq. (42) similar results are obtained. Thus for large t:

$$w_{2b} = I_1 - H_1 + I_2 - H_2 \tag{52}$$

where:

$$\begin{matrix} I_n \\ H_n \end{matrix} = \frac{p_o}{\pi m\left(\frac{c_n}{c} \mp 1\right) D_n} \int_0^\infty \frac{\sin(z \mp c_n t \pm d_n \xi^2 t)\xi}{\xi} d\xi \qquad n = 1, 2 \tag{53}$$

The constants c_2, D_2, d_2 which appear in H_2 and I_2 are defined in the same way as c_1, D_1, d_1 in eqs. (45) and (46) by changing all subscripts from 1 to 2 in these equations. The velocity c_2 is very nearly the velocity of sound in the tube wall, c_o.

The term I_1 is the deflection w_I discussed above. The H_1 term is a wave of the same velocity and shape as w_I but of smaller amplitude and traveling in the negative direction.

The H_2 and I_2 terms are also waves of the same shape as w_I, but traveling at the velocity c_2. These precursor waves are found to be very small compared to the main water hammer waves I_1, H_1 which travel at velocity c_1.

The complete deflection w is the sum of w_{2b}, eq. (52) and w_{2a}, eq. (27). The deflection w_{2a} is independent of time and its value for large values of $|z|$ may be approximated by the same kind of reasoning as used for w_{2b} for large values of t. The result is:

$$w_{2a} = \pm \frac{p_o a^2}{2Eh\left[1 + \frac{h^2}{12a^2(1-\nu^2)}\right]} \qquad \text{for } \pm z \tag{54}$$

The term $\frac{h^2}{12a^2(1-\nu^2)}$ will be neglected compared to unity since only thin-walled tubes are considered. Thus:

$$w_{2a} = \pm \frac{p_0 a^2}{2Eh} \quad \text{for } \pm z \tag{55}$$

The proceduresused to evaluate w may also be used to obtain approximations of p, v and u_z for large $|z|$ and t starting with the general expressions, eqs. (39). The results are:

$$\left.\begin{aligned} w &= \pm \frac{p_0 a^2}{2Eh} + I_1 - H_1 + I_2 - H_2 \qquad \text{for } \pm z \\ p &= \frac{p_0}{2} - \frac{2\rho_0 c_1^2}{a\left[\frac{c_1^2}{c^2} - 1\right]}(I_1 - H_1) - \frac{2\rho_0 c_2^2}{a\left[\frac{c_2^2}{c^2} - 1\right]}(I_2 - H_2) \\ v &= \frac{p_0}{2\rho_0 c} - \frac{2c_1}{a\left[\frac{c_1^2}{c^2} - 1\right]}(I_1 + H_1) - \frac{2c_2}{a\left[\frac{c_2^2}{c^2} - 1\right]}(I_2 + H_2) \\ u_z &= \mp \frac{p_0 \nu a}{2Eh} + \frac{\nu}{a\left[\frac{c_1^2}{c_0^2} - 1\right]}(I_1 - H_1) + \frac{\nu}{a\left[\frac{c_2^2}{c_0^2} - 1\right]}(I_2 - H_2) \quad \text{for } \pm z \end{aligned}\right\} \tag{56}$$

These equations lead to four wave fronts in each case as discussed for w. Numerical results for a typical case of water in a thin steel tube are shown in Fig. 5. In the present approximation for large t, the pressure and fluid velocity are again independent of r, i.e., they are constant across any section.

In order to estimate the length over which a given wave front extends, the slope w_z will be computed at the points which move with constant velocity c_1 or c_2. The difference in w before and after a wave front divided by the slope at the wave front point will be indicative of the length of a wave front.

The double integral for w_z is derived from w, eq. (20). The first integral may be evaluated and the result folded by the same procedures used for w. For a point moving so that $z = c_1 t$:

$$w_z = \frac{p_0}{\pi m}\int_0^\infty \frac{\xi \cos \xi c_1 t}{f(\xi)} d\xi + \frac{p_0}{\pi m}\int_0^\infty \sum_{\Omega_n}\left[\frac{\xi \cos(c_1\xi - \Omega)t}{\left(\frac{\Omega}{c} - \xi\right)D} - \frac{\xi\cos(c_1\xi + \Omega)t}{\left(\frac{\Omega}{c} + \xi\right)D}\right] d\xi \tag{57}$$

The function $\frac{\xi}{f(\xi)}$ is bounded for all ξ and the first integral in eq. (57) is $0\left(\frac{1}{t}\right)$ by the Riemann-Lebesgue lemmas [16] p. 172.

In the second integral the integrand is also bounded. This integral may be approximated by the method of stationary phase. The dominant part of the integral will come from the vicinity of stationary phase points which are defined by:

$$\frac{d}{d\xi}\left(\xi c_1 \pm \Omega\right) = 0 \qquad \text{i.e.} \qquad \frac{d\Omega}{d\xi} = \mp c_1 \tag{58}$$

In general such points give a contribution which is $0\left(\frac{1}{\sqrt{t}}\right)$, but if:

$$\frac{d^2}{d\xi^2}\left(\xi c_1 \pm \Omega\right) = 0 \qquad \text{i.e.} \qquad \frac{d^2\Omega}{d\xi^2} = 0 \tag{59}$$

then the contribution to the integral from the vicinity of this point is $0\left(\frac{1}{\sqrt[3]{t}}\right)$, [5] p. 40. Eqs. (58) and (59) are both satisfied on the Ω_1 curve at $\xi = 0$. Hence this contribution will predominate over all others for large values of t.

To evaluate the dominant term, the expression for Ω_1, eq. (46), is substituted in eq. (57). Hence for large t:

$$w_z = \frac{p_o}{\pi m}\int_0^\infty \frac{\cos(d_1 \xi^3 t)}{\left(\frac{c_1}{c}-1\right)D_1}\, d\xi \tag{60}$$

Changing the variable of integration to $\eta = \xi\left(\sqrt[3]{d_1 t}\right)$:

$$w_z = \frac{p_o}{\pi m\left(\frac{c_1}{c}-1\right)D_1 \sqrt[3]{d_1 t}}\int_0^\infty \cos\eta^3\, d\eta \tag{61}$$

This integral (61) is known, [5] p. 41. Hence:

$$w_z = \frac{p_o\,\Gamma\left(\frac{1}{3}\right)\sin\frac{\pi}{3}}{3\pi m\left(\frac{c_1}{c}-1\right)D_1 \sqrt[3]{d_1 t}} \tag{62}$$

Now the change of w from a location far behind the point $z = c_1 t$ to a location far ahead of the point $z = c_1 t$ is by eq. (49):

$$\Delta w = \frac{p_0}{m\left(\frac{c_1}{c}-1\right)D_1} \tag{63}$$

Defining the "nominal length" L_1 of the wave front to be Δw divided by w_z:

$$L_1 = \frac{3\pi \sqrt[3]{d_1 t}}{\Gamma\left(\frac{1}{3}\right)\sin\frac{\pi}{3}} \tag{64}$$

Computation of the "nominal length", L_2, of the wave fronts moving at velocity c_2 results in the same formula with d_1 replaced by d_2.

Some numerical values of L_1 and L_2 are given in Table 1. It may be noted that although the length of the wave front is long compared to the diameter of the pipe it is short compared to the distance from the origin that the wave has traveled.

The above computations of the lengths of wave fronts hold also for p, v and u_z because they are described by the same integrals, I_1, H_1, I_2, H_2.

4. An Extension of Joukowsky's Method

Some of the results derived by use of integral transforms may also be derived by a much simpler procedure which is an extension of Joukowsky's method. The extension consists of taking into account the longitudinal stresses and longitudinal inertia of the pipe wall. This makes possible the precursor type wave which is essentially a tension wave in the pipe wall.

The following derivation follows the general plan of Joukowsky's paper [8]. The pressure and axial velocity of the fluid are assumed to be uniform over any cross-section of the pipe. The fluid is assumed compressible, its bulk modulus being K. The equation of continuity may then be written in the form:

$$-\frac{\partial v}{\partial z} = \frac{1}{K}\frac{\partial p}{\partial t} + \frac{2}{a}\frac{\partial w}{\partial z} \tag{65}$$

The equations of motion of the fluid reduce to:

$$\rho_0 \frac{\partial v}{\partial t} = -\frac{\partial p}{\partial z} \tag{66}$$

So far, these are the equations usually used. The new element is the use of the equations of motion of the tube considered as an elastic membrane instead of a ring of no mass. These equations are derived from eqs. (5) and (6) by dropping all terms arising from bending stiffness and rotatory inertia and substituting p for q:

$$\frac{w}{a^2} + \frac{\nu}{a} u_z = \frac{p}{mc_0^2} \tag{67}$$

$$\frac{\ddot{u}}{c_0^2} - u_{zz} - \frac{\nu}{a} w_z = 0 \tag{68}$$

In eq. (67) the term $\ddot{w}$ has been omitted because the effect of radial inertia of the tube and of the fluid is neglected as in the Joukowsky theory.

No attempt will be made to solve the equations (65), (66), (67) and (68) in general. It is sufficient for the purpose at hand to show that the system permits solutions which are waves of arbitrary shape moving at either of the velocities c_1 or c_2 without dispersion. A solution is assumed of the form:

$$\left.\begin{aligned} w &= w' f(z + \bar{c} t) \\ p &= p' f(z + \bar{c} t) \\ v &= v' f(z + \bar{c} t) \\ u_z &= u'_z f(z + \bar{c} t) \\ u_t &= u'_t f(z + \bar{c} t) \end{aligned}\right\} \tag{69}$$

where the primed quantities are constants and $\bar{c}$ is an unknown constant velocity. The constants u'_t, u'_z and $\bar{c}$ are related by the requirement:

$$\frac{d}{dt} u_z = \frac{d}{dz} u_t \tag{70}$$

Substituting the assumed form (69) into eqs. (65), (66), (67), (68) and (70), there result five linear homogeneous equations involving the five primed constants. In order that these equations have a solution different from zero it is necessary for the determinant of the coefficients to be zero. This condition yields an equation

involving $\bar{c}$ which can be satisfied only if $\bar{c}$ takes on the values $\pm c_1$ or $\pm c_2$. Thus the velocities at which waves may travel without dispersion in the present considera-tion are the same as the principal velocities derived in the integral transform method.

A further point of interest is that the ratio of the change in w to the change in p across a wave front is the same in the extended Joukowsky theory and the transform solution. The same is true of any two of the quantities w, p, v, u_z, etc. This is shown by solving for the ratios of interest in each theory with $\bar{c} = c_1$ or c_2 as the case may be.

These results are of interest because they indicate the significance of the assumptions made in the Joukowsky theory. It shows that c_1 differs from c_e because the Joukowsky theory neglects longitudinal stresses and longitudinal inertia of the pipe. It also shows that the dispersion predicted by the transform method is due to the effects of the bending stiffness and of the radial inertia of the pipe and fluid. The effect of radial inertia is much more important. After the wave disperses somewhat, bending effects are negligible.

5. Other Initial Conditions

The integral transform method may be applied to certain other initial conditions beside those assumed in Sections 2 and 3. The problem and results shown in Fig. 6 illustrate a second example. The initial conditions are:

$$v = v_o \text{ for } z < 0 \qquad\qquad v = -v_o \text{ for } z > 0$$

$$p = u = w = u_t = w_t = 0 \text{ for all } z.$$

Under these conditions each half of the pipe simulates the water hammer problem in which an established flow is suddenly cut off by an instantaneous valve closure. Since by symmetry no fluid crosses the plane z = 0, each half of the pipe acts as a closed valve as far as the other half is concerned.

The magnitudes and velocities of the waves may also be established on the basis of the simplified procedure of Section 4. In general, this will be the simpler

method. It does not predict dispersion, but it may be assumed that the dispersion characteristics are the same in any case.

6. Conclusions

The theory developed herein shows the following results which are not contained in the usual theory:

1. If the initial conditions specify a sharp wave front of pressure, this front will gradually disperse over a finite length as it proceeds. This dispersion continues indefinitely so that no steady wave shape is ever reached.

2. Two waves will be generated in general; one which is similar to the Joukowsky result and one which is essentially a tension wave in the pipe wall. The pressure change due to this precursor is very small.

3. After a sufficiently long time has elapsed since the generation of a wave there will be one point in each wave front which has a constant deflection and fluid pressure and moves at a constant velocity. This velocity is very close to the Joukowsky velocity for the main pressure wave. For the precursor this velocity is slightly less than the velocity of sound in the pipe wall.

4. The bending stresses in the pipe wall have very little effect on the characteristic velocities and the rates of dispersion after a sufficient time has elapsed. Such stresses can only be significant in the early stages of the motion in regions where the wave front is sharp so that the variation of pressure and deflection is very rapid along the length of the pipe.

5. The relations between fluid velocity, pressure and hoop stress given by the Joukowsky theory are substantially correct, but there are also longitudinal stresses in the pipe not predicted by the usual theory.

[2]

6. Available experimental data is in very close agreement with the predictions of the Joukowsky theory as to the pressure rise and velocity of the main pressure wave. Inasmuch as the analysis presented in this paper gives essentially the same results as the Joukowsky theory for these items, it is confirmed experimentally. However, there is no experimental data available on the precursor wave or the rate of dispersion predicted herein.

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APPENDIX

Expansion of Ω_1 and Ω_2 About $\xi = 0$

The values Ω_1, Ω_2 are defined as the lowest two roots of eq. (25). From the numerical results represented by Fig. 2, it is expected that both Ω_1 and Ω_2 approach zero as ξ approaches zero. Hence, Ω_1 and Ω_2 have an expansion of the form:

$$\Omega = \bar{c}\,\xi - d\,\xi^3 + \dots \tag{71}$$

where only odd powers occur because the Ω_n are odd functions of ξ. The values of $\bar{c}$ and d are determined by substituting this expansion in eq. (25) and equating the coefficient of each power of ξ appearing to zero.

The terms $\frac{1}{N}$ and Δ in the transcendental equation (25) are given in full by eqs. (23) and (24). Expanding the Bessel functions, Δ may be written in the form:

$$\Delta = \frac{\left[1 - \frac{a^2}{4}\left(\frac{\Omega^2}{c^2} - \xi^2\right) + \frac{a^4}{64}\left(\frac{\Omega^2}{c^2} - \xi^2\right)^2 + \dots\right]}{\frac{a^2}{4}\left(\frac{\Omega^2}{c^2} - \xi^2\right)\left[1 - \frac{a^2}{8}\left(\frac{\Omega^2}{c^2} - \xi^2\right) + \frac{a^4}{192}\left(\frac{\Omega^2}{c^2} - \xi^2\right)^2 + \dots\right]} \tag{72}$$

In order to be able to clear eq. (25) of fractions, the quotient of the two infinite series in eq. (72) is written as a single series:

$$\Delta = \frac{1}{\frac{a^2}{4}\left(\frac{\Omega^2}{c^2}-\xi^2\right)}\left[1-\frac{a^2}{8}\left(\frac{\Omega^2}{c^2}-\xi^2\right)-\frac{a^4}{192}\left(\frac{\Omega^2}{c^2}-\xi^2\right)^2+\cdots\right] \tag{73}$$

Using this value the transcendental eq. (25) may be put in the form:

$$\begin{aligned}\frac{2\rho_0\Omega^2}{ma}\left(\frac{\Omega^2}{c_0^2}-\xi^2\right)&\left[1-\frac{a^2}{8}\left(\frac{\Omega^2}{c^2}-\xi^2\right)-\frac{a^4}{192}\left(\frac{\Omega^2}{c^2}-\xi^2\right)^2+\cdots\right]+\\ &+\Omega^2\left(\frac{\Omega^2}{c^2}-\xi^2\right)\left(\frac{\Omega^2}{c_0^2}-\xi^2\right)\left[-1-\frac{h^2\xi^2}{12}\right]+\\ &+\left(\frac{\Omega^2}{c^2}-\xi^2\right)\left(\frac{\Omega^2}{c_0^2}-\xi^2\right)\frac{h^2c_0^2}{12}\left[\frac{1}{a^4}+\xi^4\right]+\\ &+\left(\frac{\Omega^2}{c^2}-\xi^2\right)\frac{1}{a^2}\left[\Omega^2-c_0^2(1-\nu^2)\xi^2+\frac{\nu h^2c_0^2\xi^4}{6}+\frac{c_0^2h^4\xi^6}{144}\right]=0\end{aligned} \tag{74}$$

Now the assumed expansion given by eq. (71) is substituted above for Ω. The result is a series of the form:

$$b_1\xi^4+b_2\xi^6+b_3\xi^8+\cdots = 0 \tag{75}$$

Each coefficient b_n of this series must be zero since the value Ω is a root of the transcendental eq. (25). Equating b_1 to zero and neglecting $\frac{h^2}{a^2}$ compared to unity yields an equation involving $\bar{c}$ but not d. This is interpreted as an equation on the yet unknown $\bar{c}$. The roots are:

$$c_1, c_2 = c\left[\frac{2AR+R+R^2(1-\nu^2)\mp\sqrt{[2AR+R+R^2(1-\nu^2)]^2-4R^2(1-\nu^2)(2A+R)}}{2(2A+R)}\right] \tag{76}$$

where

$$R = \frac{c_0^2}{c^2} \qquad\qquad A = \frac{\rho_0 a}{m} \tag{77}$$

Equating b_2 to zero yields an equation involving both $\bar{c}$ and d. This equation is considered to determine two values of d, one for $\bar{c} = c_1$ and one for $\bar{c} = c_2$. The solution of the equation $b_2 = 0$ for d yields:

$$d = ca^2\left[\frac{(A+4)\left[\left(\frac{\bar{c}}{c}\right)^5-\left(\frac{\bar{c}}{c}\right)^3(1+R)+\left(\frac{\bar{c}}{c}\right)R\right]-\left(\frac{\bar{c}}{c}\right)\frac{h^2}{12a^2}\left(1+2R^2\nu\right)}{-16\left(\frac{\bar{c}}{c}\right)^2(2A+R)+8R(2A+1)+8R^2(1-\nu^2)}\right] \tag{78}$$

Table I. Nominal Lengths of Wave Fronts for Typical Example

	Time Seconds	Nominal length of wave front. Feet	Distance of wave front from origin. Feet
Primary Wave	1	19.1	3,218
	5	32.6	16,091
Precursor Wave	1	30.0	17,325
	5	51.3	86,640

Assumed Values		Derived Values
Radius of pipe:	a = 1 foot	$\frac{a}{h} = 62.752$
Velocity of sound in water:	c = 5,000 ft/sec	R = 12.478
Density of water:	$\rho_o = 1.94$ slugs/ft^3	$\frac{c_1}{c} = 0.64365$
Modulus of elasticity of steel:	$E = 30 \times 10^6$ psi	$\frac{c_2}{c} = 3.4656$
Poisson's ratio for steel:	$\nu = 0.30$	$\frac{c_o}{c} = 3.5324$
Specific weight of steel:	$\gamma_s = 490$ lbs/ft^3	$\frac{c_e}{c} = 0.64429$
Ratio, $A = \frac{\rho_o a}{m}$:	A = 8	

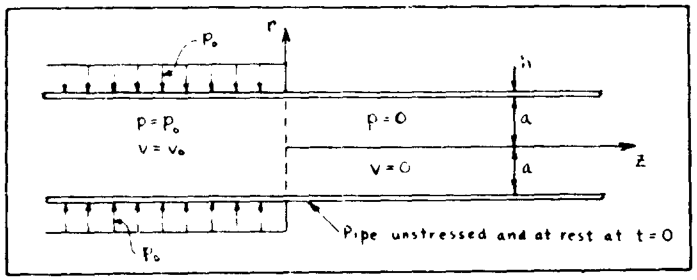

Fig.1

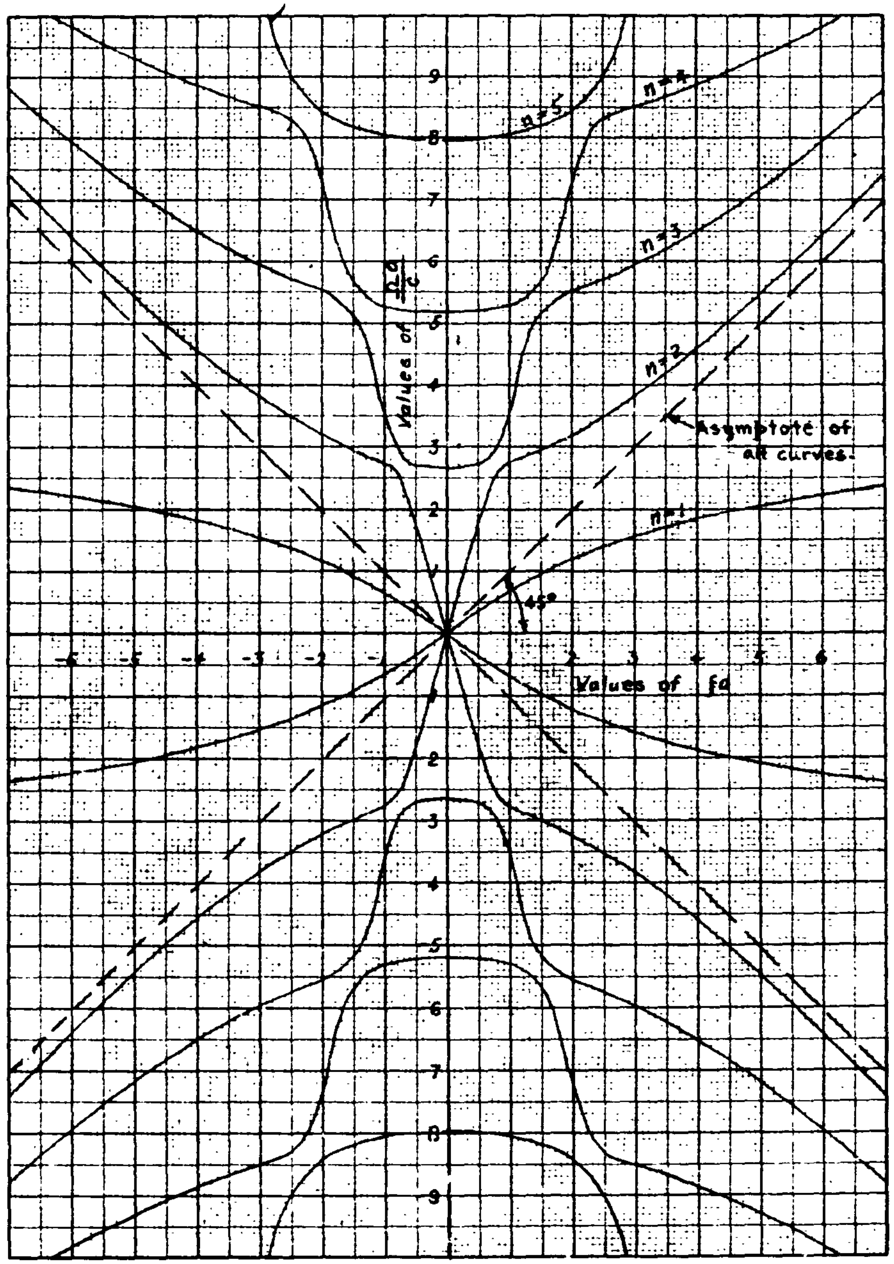

Fig.2

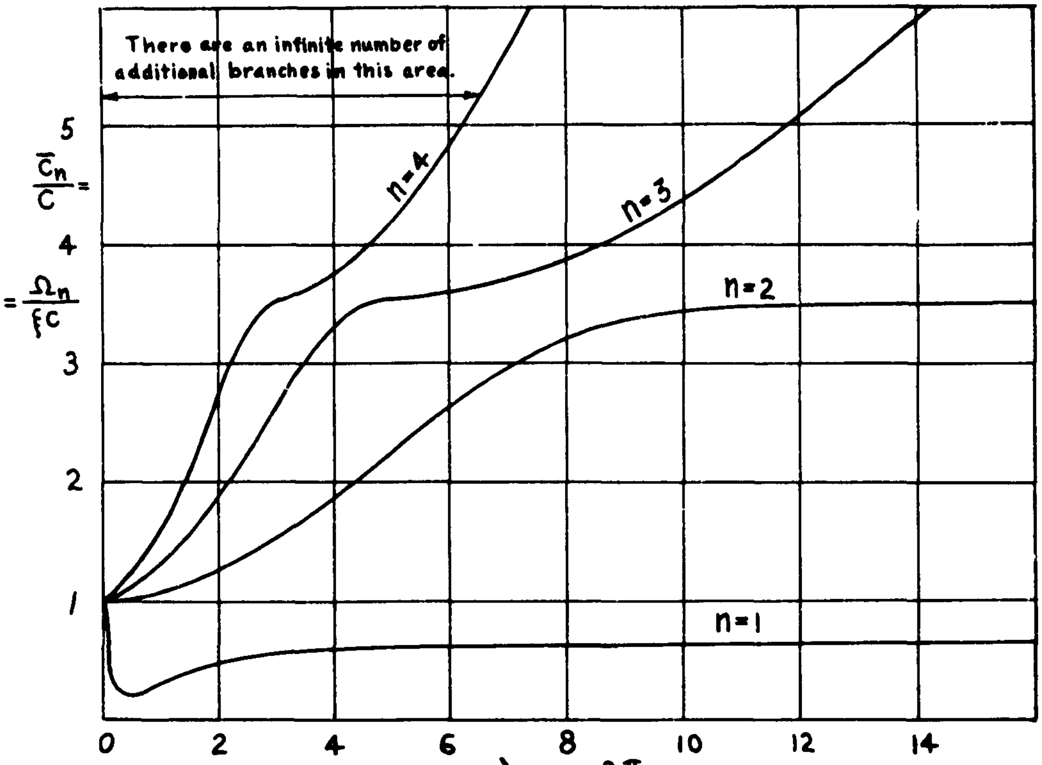

FIG. 3

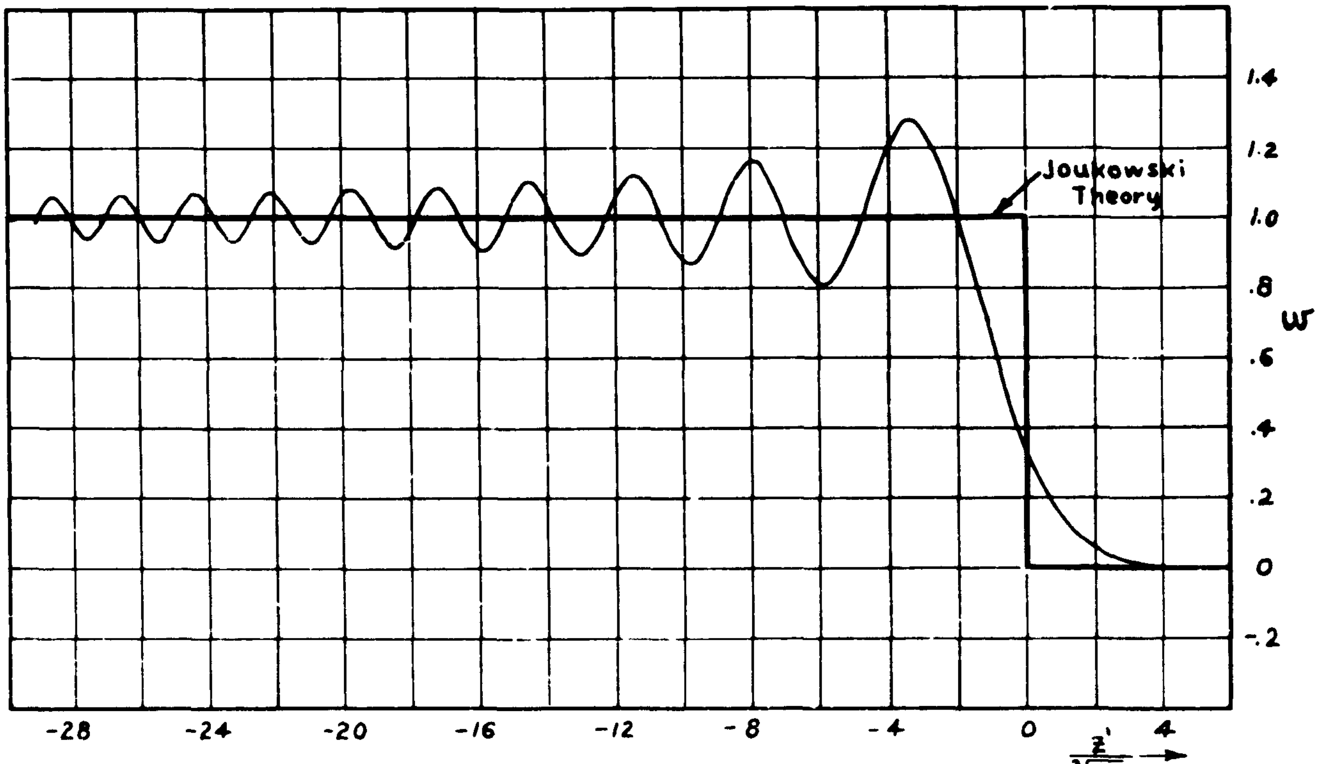

Fig. 4 Form of Typical Wave Front

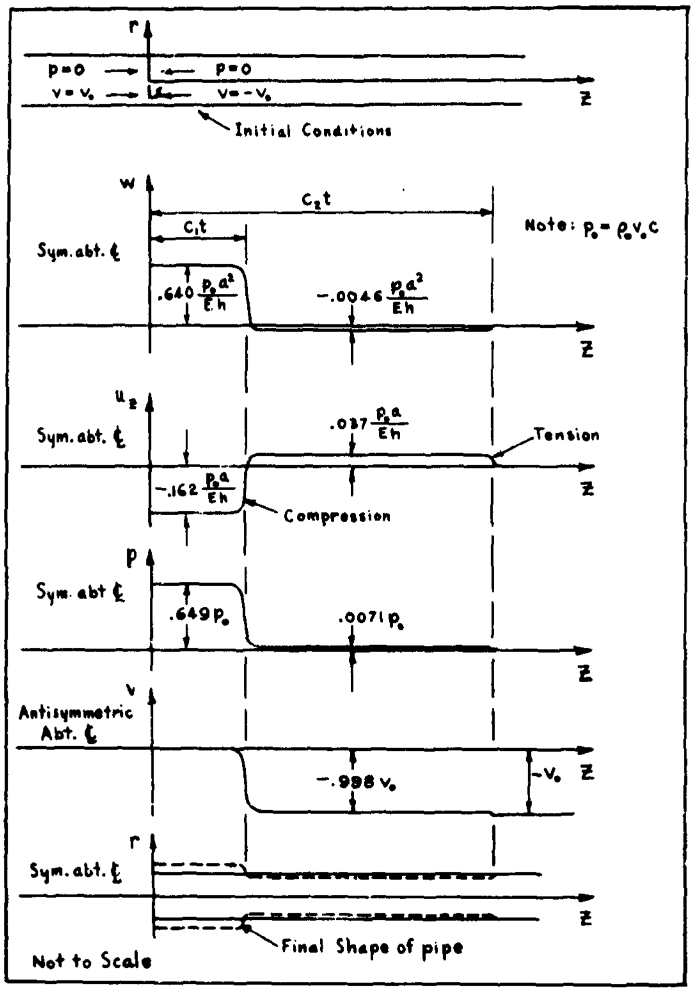

Fig. 6. Values of w, u_z, p and v for a steel tube filled with water, $\frac{a}{h} = 62.8$. Case of sudden stoppage of flow.

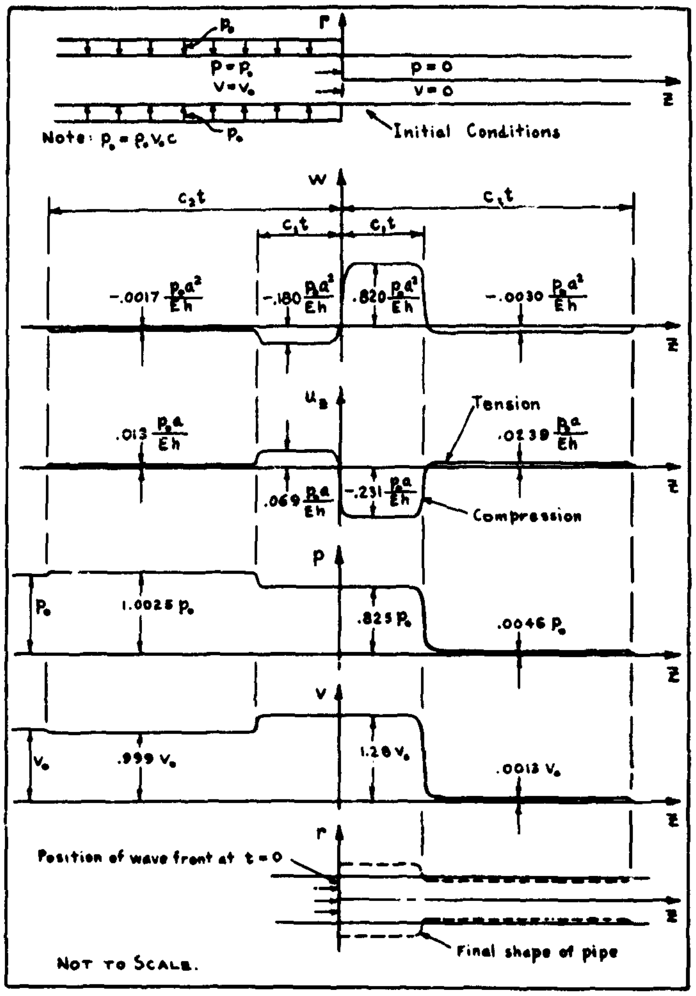

Fig. 5. Values of w, u_z, p and v for a steel tube filled with water, $\frac{a}{h} = 62.8$. Case of initiation of flow.

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